Mathematical Institute of Charles University

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Bifurcation analysis of viscoelastic flows using deflation method

Mathematical Aspects of Fluid Flows

Kácov 2024

Deflated Continuation Method

- Deflation techniques[1]
  - Goal: find multiple distinct solutions for stationary nonlinear equations
  - Idea: systematically modify a nonlinear problem to ensure that Newton's method will not converge to known roots
- Continuation methods
  - Goal: extend solution branches for other values of λ

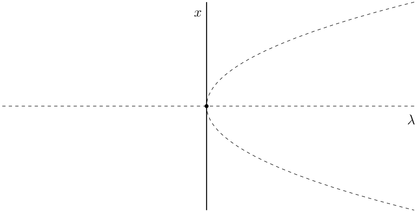

Figure: Pitchfork bifurcation $F(x, \lambda) = \lambda x - x^3 = 0$.

[1] Farrell, Birkisson, and Funke, "Deflation Techniques for Finding Distinct Solutions of Nonlinear Partial Differential Equations".

Deflation techniques

- Problem $F(u) = 0$, where $F \in C^1(X, Y)$
- Known solution $u_r \in X$, $F(u_r) = 0$, $F_u(u_r)$ non-singular
- Deflated problem $G(u) = M(u, u_r)F(u) = 0$ satisfying following properties
  1. if $u \neq u_r$ then $G(u) = 0 \iff F(u) = 0$
  2. Newton's method does not converge to u_r.
- The shifted deflation operator

$$M(u, u_r) = \left(\frac{1}{||u - u_r||^p} + \alpha \right) \mathbb{I}$$

- Solve the deflated problem with the same initial guesses
- Repeat the procedure until no more solutions are found

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Viscoelastic fluid

- Dimensionless equations for steady state flows

$$0 = \operatorname{div} \boldsymbol{v} \qquad 0 = -\left(\boldsymbol{v} \bullet \nabla\right) \boldsymbol{v} + \operatorname{div} \mathbb{T} \qquad 0 = -\left(\boldsymbol{v} \bullet \nabla\right) \mathbb{B}_{\kappa_{p(t)}} + \left(\nabla \boldsymbol{v}\right) \mathbb{B}_{\kappa_{p(t)}} + \mathbb{B}_{\kappa_{p(t)}} \left(\nabla \boldsymbol{v}\right)^{\top} - \frac{1}{\mathrm{Wi}} \boldsymbol{f}(\mathbb{B}_{\kappa_{p(t)}})$$

- $\mathbb{B}_{\kappa_{p(t)}}$ left Cauchy-Green tensor associated with the elastic part of the fluid response (**symmetric positive definite**)

- Two models
  - **FENE-CR** model

$$f(\mathbb{B}_{\kappa_{p(t)}}) = \frac{L^2}{L^2 - \operatorname{Tr} \mathbb{B}_{\kappa_{p(t)}}} \left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right) \qquad \mathbb{T} = -p\mathbb{I} + 2\frac{\beta}{\mathrm{Re}}\mathbb{D} + \frac{1-\beta}{\mathrm{Re}\mathrm{Wi}} \frac{L^2}{L^2 - \operatorname{Tr} \mathbb{B}_{\kappa_{p(t)}}} \left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right)$$

  - **Giesekus** model

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- Parameters: the Reynolds number Re, the Weissenberg number Wi and the solvent viscosity ratio β.

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Numerical Method

- Finite element method
- Straightforward discretization
  - $\mathbb{B}_{\kappa_{p(t)}}$ loses positive definiteness during computation
  - Continuation and deflation method often fail
- Reformulation of the equations based on the symmetric square root[2]

$$\mathbb{B}_{\kappa_{p(t)}} = \mathbb{C}^2$$

- Finite element stabilization techniques[3] (DEVSS–TG, SUPG)
- The norm in the deflation operator

$$\|v - v_r\| + \|\nabla v - \nabla v_r\|.$$

[2] Balci et al., "Symmetric factorization of the conformation tensor in viscoelastic fluid models".

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Viscoelastic fluid

- Planar sudden expansion with expansion ratio 1:4 where symmetry breaking bifurcation occurs above critical Reynolds number

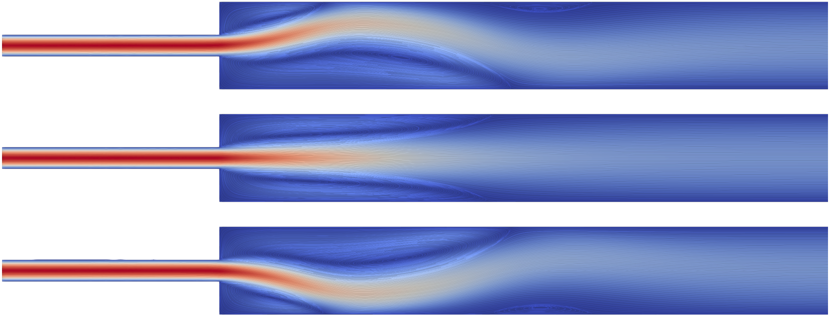

Figure: FENE–CR model, $\mathrm{Re} = 80$, $\mathrm{Wi} = 2$, $\beta = 0.5$, $L^2 = 100$

Sudden expansion[4] with expansion ratio 1:4

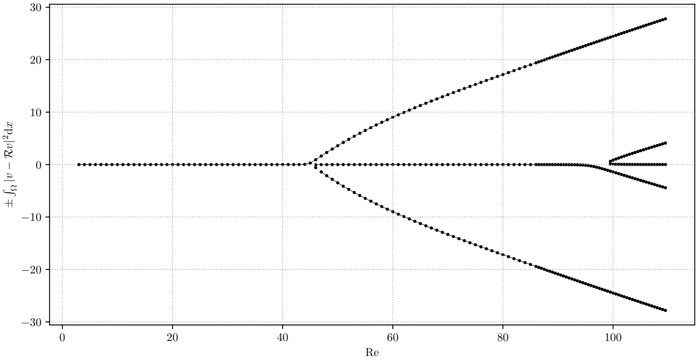

Figure: Bifurcation diagram, FENE–CR model, $\mathrm{Wi} = 2$, $L^2 = 100$, $\beta = 0.5$

[4] Rocha, Poole, and Oliveira, "Bifurcation phenomena in viscoelastic flows through a symmetric 1:4 expansion".

Sudden expansion with expansion ratio 1:4

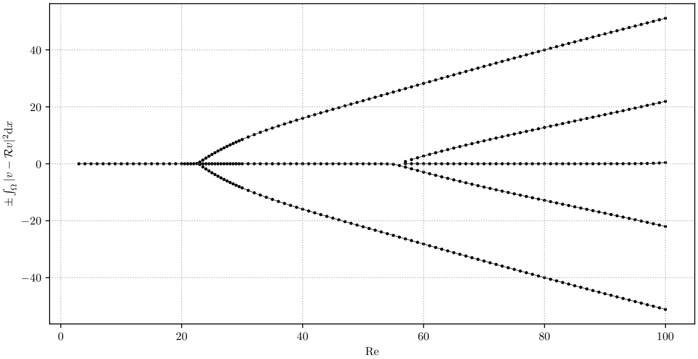

Figure: Bifurcation diagram, Giesekus model, $\mathrm{Wi} = 5$, $\alpha = 0.5$, $\beta = 0.5$