Bifurcation analysis of viscoelastic flows using deflation method

Ladislav Trnka

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

Overview

We study bifurcation behaviour of viscoelastic fluid flows. Our goal is to find multiple steady-state/stationary flows using the deflated continuation method [1] in two geometries where symmetry-breaking bifurcations occur at high enough system parameters. We use the first setting, flows of FENE-CR fluid in a cross-slot geometry [2], as a benchmark to show the viability of the deflated continuation approach. In the second setting, flows of FENE-CR [3] and Giesekus fluids in a sudden expansion geometry, we investigate the impact of the elastic effects of different viscoelastic fluid models on the onset of the bifurcations. The problems are solved by finite element method. To alleviate numerical breakdowns and exclude spurious solutions from potential convergence of Newton's method, we use the logarithm formulation [4] with the DEVSS-TG/SUPG stabilisation of the governing equations [5]. The numerical methods are implemented using Firedrake and Defcon library.

Deflated Continuation Method

To construct bifurcation diagrams, we use the deflated continuation method. The method combines deflation techniques (find multiple solutions at fixed parameter values) and continuation methods (extend solution branches), see [1, 6].

Deflation techniques aim to systematically modify a nonlinear problem so that Newton's method does not converge to the already known solutions. Instead of that, Newton's method is forced to search for previously unknown solutions. The deflated continuation method has several advantages:

- disconnected solution branches can be found,
- good scalability with respect to problem dimension (no computation of subproblems required).

Suppose a parameter-dependent stationary nonlinear problem which permits multiple solutions

$$F(u,\lambda) = 0.$$

Suppose further that we know k solutions u_r at one specific $\lambda = \tilde{\lambda}$. The goal is to find another one. The essence of the deflation techniques is to solve the modified problem $G(u,\tilde{\lambda}) = M(u,u_r)F(u,\tilde{\lambda}) = 0$, where $M(u,u_r)$ is the deflation operator and $\tilde{\lambda}$ is fixed. The modified problem satisfies following properties:

- if $u \neq u_r$ then $G(u,\tilde{\lambda}) = 0 \iff F(u,\tilde{\lambda}) = 0$,
- Newton's method does not converge to u_r.

Farrell et al. [6] proposed the shifted deflation operator in the form

$$M(u,u_r) = \prod_{r=1}^{k} \left(\frac{1}{\|u-u_r\|^p} + \alpha \right) \mathbb{I},$$

where p is the power and α is the shift. In this work, we choose $p = 2$, $\alpha = 1$ and the norm in the deflation operator in the form

$$\|\mathbf{v}-\mathbf{v}_r\|_2^2 + \|\nabla\mathbf{v}-\nabla\mathbf{v}_r\|_2^2 + \|\mathbb{B}_{\kappa_{p(t)}} - \mathbb{B}^r_{\kappa_{p(t)}}\|_2^2.$$

By solving the modified problem, we try to find as many solutions as possible for the fixed $\lambda = \tilde{\lambda}$. Then we extend the solution branches for other values of λ using the classical (natural) continuation method.

References

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Governing Equations and Logarithm Reformulation

We consider two types of incompressible viscoelastic fluids described by the FENE–CR model and the Giesekus model. The dimensionless equations for steady state flows ($\mathbf{v}$ velocity, p pressure, $\mathbb{B}_{\kappa_{p(t)}}$ extra stress tensor) take the form

$$0 = \operatorname{div}\mathbf{v}, \qquad 0 = -(\mathbf{v}\bullet\nabla)\mathbf{v} + \operatorname{div}\mathbb{T}, \qquad 0 = -(\mathbf{v}\bullet\nabla)\mathbb{B}_{\kappa_{p(t)}} + (\nabla\mathbf{v})\mathbb{B}_{\kappa_{p(t)}} + \mathbb{B}_{\kappa_{p(t)}}(\nabla\mathbf{v})^\top - \frac{1}{\mathrm{Wi}}\boldsymbol{f}\left(\mathbb{B}_{\kappa_{p(t)}}\right),$$

where $\mathbb{B}_{\kappa_{p(t)}}$ is the extra stress tensor (symmetric positive definite) associated with the elastic part of the fluid response. The Cauchy stress tensor $\mathbb{T}$ and $f(\mathbb{B}_{\kappa_{p(t)}})$ are given by

FENE-CR model

$$\boldsymbol{f}\left(\mathbb{B}_{\kappa_{p(t)}}\right) = \frac{b}{b-\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}}}\left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right),$$

$$\mathbb{T} = -p\mathbb{I} + 2\frac{\beta}{\mathrm{Re}}\mathbb{D} + \frac{1-\beta}{\mathrm{ReWi}}\frac{b}{b-\operatorname{Tr}\mathbb{B}_{\kappa_{p(t)}}}\left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right),$$

Giesekus model

$$\boldsymbol{f}\left(\mathbb{B}_{\kappa_{p(t)}}\right) = \alpha\mathbb{B}^2_{\kappa_{p(t)}} + (1-2\alpha)\mathbb{B}_{\kappa_{p(t)}} - (1-\alpha)\mathbb{I},$$

$$\mathbb{T} = -p\mathbb{I} + 2\frac{\beta}{\mathrm{Re}}\mathbb{D} + \frac{1-\beta}{\mathrm{ReWi}}\left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I}\right),$$

where b denotes the extensibility parameter and α the mobility parameter. The dimensionless numbers are the Reynolds number Re, the Weissenberg number Wi and the solvent viscosity ratio β.

In practice, a straightforward discretisation of the governing equations by the finite element method lead to numerical difficulties during continuation and deflation (detection of spurious steady-state solutions violating the positive definiteness of $\mathbb{B}_{\kappa_{p(t)}}$, numerical breakdowns at high enough Weissenberg numbers). To alleviate numerical breakdowns and and preserve positive definiteness of $\mathbb{B}_{\kappa_{p(t)}}$, we solve reformulated equations for the matrix logarithm of $\mathbb{B}_{\kappa_{p(t)}}$,

$$\mathbb{S} = \ln(\mathbb{B}_{\kappa_{p(t)}}), \qquad \nabla\mathbf{v} = \Omega + \mathbb{E} + \mathbb{N}\mathbb{B}^{-1}_{\kappa_{p(t)}}, \qquad \frac{\mathrm{d}\mathbb{S}}{\mathrm{d}t} = \Omega\mathbb{S} - \mathbb{S}\Omega + 2\mathbb{E} - \frac{1}{\mathrm{Wi}}\mathfrak{f}\left(\mathbb{B}_{\kappa_{p(t)}}\right)\mathbb{B}^{-1}_{\kappa_{p(t)}},$$

for details, see [4]; and we employ DEVSS–TG numerical scheme with SUPG stabilisation of the convection term in the constitutive equation, see [5].

Cross-slot Geometry

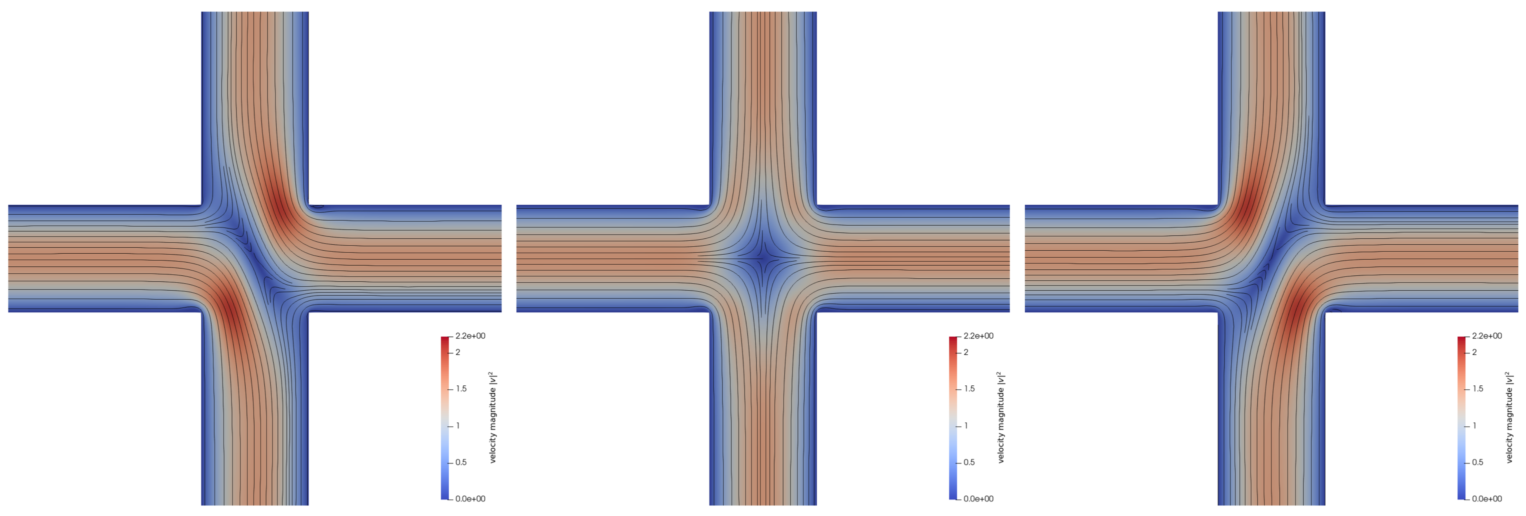

Figure 1: Multiple steady-state flows of **FENE-CR fluid**, Wi = 0.5, Re = 0.01, $b = 100$, $\beta = 0.1$

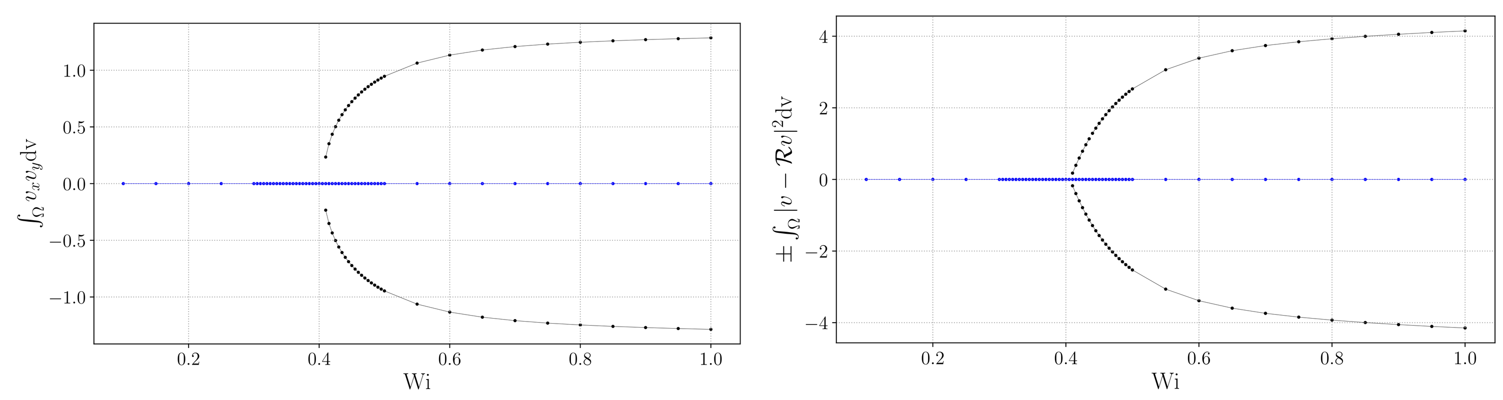

Figure 2: Bifurcation diagrams (bifurcation parameter is the Weissenberg number), **FENE-CR model**, Re = 0.01, $b = 100$, $\beta = 0.1$, this work: $\mathrm{Wi}_{\mathrm{crit}} \in (0.400, 0.405)$, previous study [2]: $\mathrm{Wi}_{\mathrm{crit}} \approx 0.46$

We have obtained qualitatively the same results as in the previous study [2]. The differences in critical values may have been caused by our particular choice of the stabilisation formulation, discretisation, and by the specific approach to numerical bifurcation analysis.

Sudden Expansion Geometry

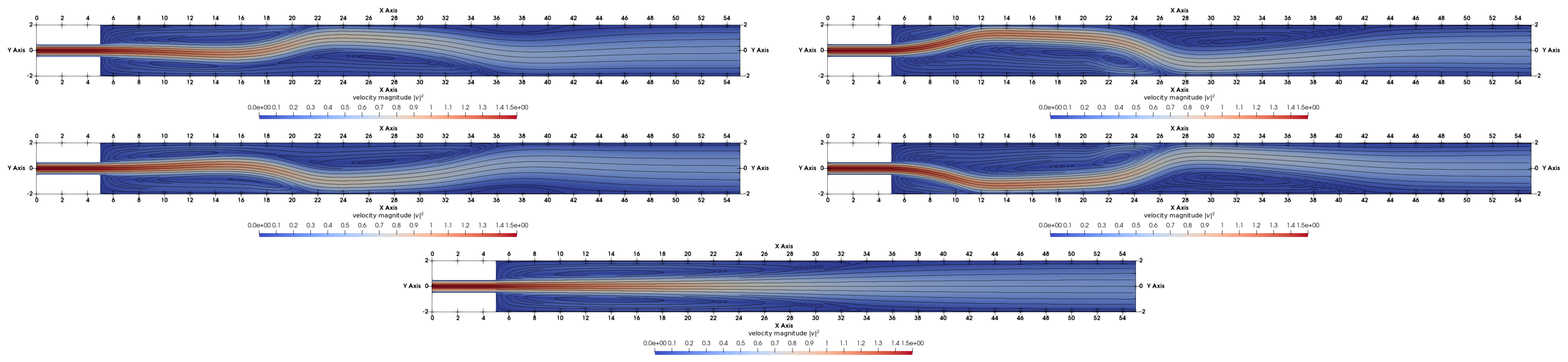

Figure 3: Multiple steady-state flows of **Giesekus fluid**, Re = 100, Wi = 5.0, $\alpha = 0.5$, $\beta = 0.5$

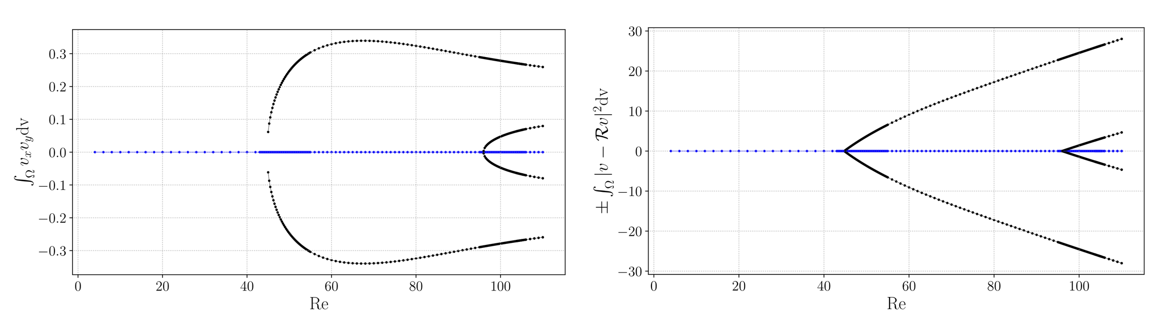

Figure 4: Bifurcation diagrams (bifurcation parameter is the Reynolds number), **FENE-CR model**, Wi = 2.0, $b = 100$, $\beta = 0.5$, identified critical values: $\mathrm{Re}_{\mathrm{crit},1} \in (44.75, 45.00)$, $\mathrm{Re}_{\mathrm{crit},2} \in (95.75, 96.00)$

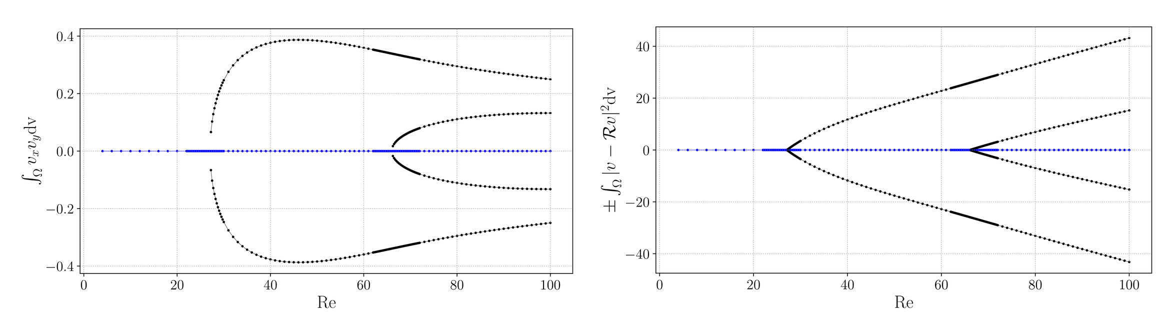

Figure 5: Bifurcation diagrams (bifurcation parameter is the Reynolds number), **Giesekus model**, Wi = 2.0, $\alpha = 0.5$, $\beta = 0.5$, identified critical values: $\mathrm{Re}_{\mathrm{crit},1} \in (27.00, 27.25)$, $\mathrm{Re}_{\mathrm{crit},2} \in (66.00, 66.25)$

Considering different viscoelastic fluid models, the elasticity of viscoelastic fluid flows can both promote and delay the onset of bifurcations in the sudden expansion channel. In particular, based on our numerical computations, symmetry-breaking bifurcations occur first for flows described by the Giesekus model, then for Newtonian fluid flows [identified critical values: $\mathrm{Re}_{\mathrm{crit},1} \in (36.00, 36.25)$, $\mathrm{Re}_{\mathrm{crit},2} \in (85.00, 85.25)$], and lastly for flows described by the FENE-CR model in the same setting.

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Contact

Ladislav Trnka, `trnka.ladis@gmail.com`