Bifurcation analysis of viscoelastic flows using deflation method

Ladislav Trnka
Mathematical Institute of Charles University

Abstract

We study the bifurcation behaviour of viscoelastic fluid flows. In particular, we investigate steady states of Giesekus and FENE-CR fluid in a planar sudden expansion geometry with expansion ratio 1:4 where symmetry-breaking bifurcation occurs above critical Reynolds number. For bifurcation analysis, we use the deflated continuation method which combines the deflation techniques (find multiple solutions at fixed parameter values) with continuation methods (extend solution branches). During continuation and deflation, the left Cauchy–Green tensor associated with the elastic part of the fluid response loses positive definiteness. To preserve its positive definiteness, we solve reformulated equations for the matrix logarithm of the left Cauchy–Green tensor. The problem is solved by finite element method. In addition, we apply finite element stabilization techniques (DEVSS-TG, SUPG) used for viscoelastic flows. The numerical methods are implemented using Firedrake and Defcon library.

Deflated Continuation Method

To construct bifurcation diagrams, we use the deflated continuation method. The method is based on deflation techniques and classical continuation method, see Farrell et al. [2015] and Farrell et al. [2016]. The idea behind deflation techniques is to systematically modify a nonlinear problem to ensure that Newton's method does not converge to known solutions in order to search for new solutions. In contrast to other methods in numerical bifurcation analysis, the deflated continuation method has several advantages:

- disconnected solution branches can be found,
- good scalability with respect to problem dimension (no computation of subproblems required),
- no special insight in choosing initial guesses is required (finding multiple solutions from the same initial guess).

Suppose a parameter-dependent stationary nonlinear problem which permits multiple solutions

$$F(u, \lambda) = 0.$$

Suppose further that we know k solutions u_r at one specific $\lambda = \hat{\lambda}$ and the goal is to find another one. The essence of the deflation techniques is to solve the deflated problem $G(u, \hat{\lambda}) = M(u, u_r)F(u, \hat{\lambda}) = 0$, where $M(u, u_r)$ is the deflation operator and $\hat{\lambda}$ is fixed. The deflation problem satisfies following properties:

- if $u \neq u_r$ then $G(u) = 0 \iff F(u) = 0$,
- Newton's method does not converge to u_r.

Farrell et al. [2015] proposed the shifted deflation operator in the form

$$M(u, u_r) = \prod_{r=1}^{k} \left(\frac{1}{||u - u_r||^p} + \alpha \right) \mathbb{I},$$

where p is the power and α is the shift (we use $p = 2$ and $\alpha = 1$). By solving the deflated problem, we try to find as many solutions as possible for the fixed $\lambda = \hat{\lambda}$. Then we extend the solution branches for other values of λ using continuation method.

References

Frank Baaijens. Mixed finite element methods for viscoelastic flow analysis: a review. *J. of Non-Newton. Fluid Mech.*, 79(2):361–385, 1998. doi: 10.1016/S0377-0257(98)00122-0.

Nusret Balci, Becca Thomases, Michael Renardy, and Charles R. Doering. Symmetric factorization of the conformation tensor in viscoelastic fluid models. *J. of Non-Newton. Fluid Mech.*, 166(11):546–553, 2011. doi: 10.1016/j.jnnfm.2011.02.008.

P. E. Farrell, Á. Birkisson, and S. W. Funke. Deflation techniques for finding distinct solutions of nonlinear partial differential equations. *SIAM J. on Scientific Computing*, 37(4):A2026–A2045, 2015. doi: 10.1137/140984798.

Patrick E. Farrell, Casper H. L. Beentjes, and Ásgeir Birkisson. The computation of disconnected bifurcation diagrams, 2016.

R. Fattal and R. Kupferman. Constitutive laws for the matrix-logarithm of the conformation tensor. *J. of Non-Newton. Fluid Mech.*, 123(2):281–285, 2004. ISSN 0377-0257. doi: 10.1016/j.jnnfm.2004.08.008.

Gerardo N. Rocha, Robert J. Poole, and Paulo J. Oliveira. Bifurcation phenomena in viscoelastic flows through a symmetric 1:4 expansion. *J. of Non-Newton. Fluid Mech.*, 141(1):1–17, 2007. doi: 10.1016/j.jnnfm.2006.08.008.

Governing Equations

We consider two types of incompressible viscoelastic fluids described by the Giesekus model and the FENE–CR model. The dimensionless equations for steady state flows (v velocity, p pressure, $\mathbb{B}_{\kappa_{p(t)}}$ extra stress tensor) take the form

$$0 = \operatorname{div} \mathbf{v}, \qquad 0 = -(\mathbf{v} \bullet \nabla)\mathbf{v} + \operatorname{div} \mathbb{T}, \qquad 0 = -(\mathbf{v} \bullet \nabla)\mathbb{B}_{\kappa_{p(t)}} + (\nabla \mathbf{v})\mathbb{B}_{\kappa_{p(t)}} + \mathbb{B}_{\kappa_{p(t)}}(\nabla \mathbf{v})^{\top} - \frac{1}{\mathrm{Wi}} \boldsymbol{f}(\mathbb{B}_{\kappa_{p(t)}}),$$

where $\mathbb{B}_{\kappa_{p(t)}}$ can be interpreted as the left Cauchy-Green tensor (**symmetric positive definite**) associated with the elastic part of the fluid response. The Cauchy stress tensor $\mathbb{T}$ and $f(\mathbb{B}_{\kappa_{p(t)}})$ are given by the models

FENE-CR model

$$\boldsymbol{f}(\mathbb{B}_{\kappa_{p(t)}}) = \frac{L^2}{L^2 - \operatorname{Tr} \mathbb{B}_{\kappa_{p(t)}}} \left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I} \right),$$

$$\mathbb{T} = -p\mathbb{I} + 2\frac{\beta}{\mathrm{Re}}\mathbb{D} + \frac{1-\beta}{\mathrm{ReWi}} \frac{L^2}{L^2 - \operatorname{Tr} \mathbb{B}_{\kappa_{p(t)}}} \left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I} \right),$$

Giesekus model

$$\boldsymbol{f}(\mathbb{B}_{\kappa_{p(t)}}) = \alpha \mathbb{B}^2_{\kappa_{p(t)}} + (1 - 2\alpha)\mathbb{B}_{\kappa_{p(t)}} - (1 - \alpha)\mathbb{I},$$

$$\mathbb{T} = -p\mathbb{I} + 2\frac{\beta}{\mathrm{Re}}\mathbb{D} + \frac{1-\beta}{\mathrm{ReWi}} \left(\mathbb{B}_{\kappa_{p(t)}} - \mathbb{I} \right),$$

where L denotes the extensibility parameter and α the mobility parameter. The dimensionless numbers are the Reynolds number Re, the Weissenberg number Wi and the solvent viscosity ratio β.

Reformulation and Numerical Method

In practice, straightforward discretization of weak formulation of the governing equations lead to numerical difficulties during continuation and deflation (high Weissenberg number problem, $\mathbb{B}_{\kappa_{p(t)}}$ loses positive definiteness). To alleviate HWNP and keep $\mathbb{B}_{\kappa_{p(t)}}$ positive definite, following reformulations of the governing equations were introduced.

log–conformation representation (Fattal and Kupferman [2004])	square root matrix transformation (Balci et al. [2011])
$\mathbb{S} = \log \mathbb{B}_{\kappa_{p(t)}}$	$\mathbb{V} = \sqrt{\mathbb{B}_{\kappa_{p(t)}}}$

Steady states (v,p,$\mathbb{S}$) are directly computed by solving stationary equations using finite element method. The velocity and pressure fields are approximated by Taylor–Hood element and components of tensorial quantity $\mathbb{S}$ are approximated by continuous piecewise linear elements. To improve numerical stability, we employ DEVSS–TG numerical scheme with SUPG stabilization of the convection term in the constitutive equation, see Baaijens [1998]. We construct bifurcation diagrams using the deflated continuation method. The norm in the deflation operator is chosen as

$$\|v - v_r\|_2^2 + \|\nabla v - \nabla v_r\|_2^2 + \|\mathbb{B}_{\kappa_{p(t)}} - \mathbb{B}_{\kappa_{p(t)},r}\|_2^2.$$

We use the deflated continuation algorithm implemented in Defcon library[a] for Firedrake.

[a] https://bitbucket.org/pefarrell/defcon/src/master/

Sudden Expansion

In a planar sudden expansion with expansion ratio 1:4, symmetry-breaking bifurcations occur above a critical Reynolds number $\mathrm{Re}_{\mathrm{crit}}$, see Figure 1. Imposed boundary conditions are: fully developed flow on the inlet ($v = (1.5(1 - 4y^2), 0)$ and numerically solved $\mathbb{S}$), traction-free condition on the outlet and no-slip condition on the walls.

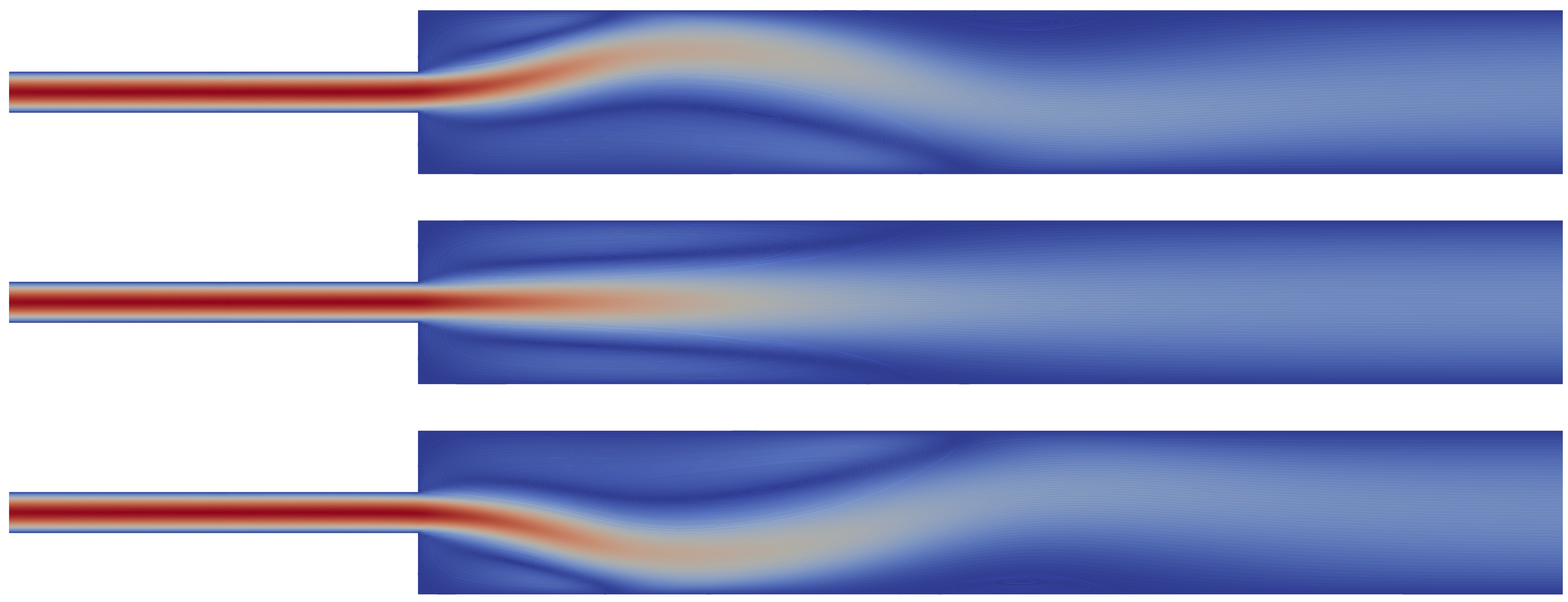

Figure 1: FENE–CR model (velocity magnitude), $\mathrm{Re} = 80, \mathrm{Wi} = 2, \beta = 0.5, L^2 = 100$

For the FENE–CR model ($\mathrm{Wi} = 2$, $\beta = 0.5$, $L^2 = 100$), Rocha et al. [2007] identified the critical Reynolds number about 46. We obtain the same results with a different numerical approach, see numerical method above. The critical Reynolds number for the Giesekus fluid ($\mathrm{Wi} = 5, \alpha = 0.5, \beta = 0.5$) is $\mathrm{Re}_{\mathrm{crit}} \approx 23$, which is less than the critical Reynolds number for flows given by the Navier–Stokes equations ($\mathrm{Re}_{\mathrm{crit}} \approx 36$) in the same setting.

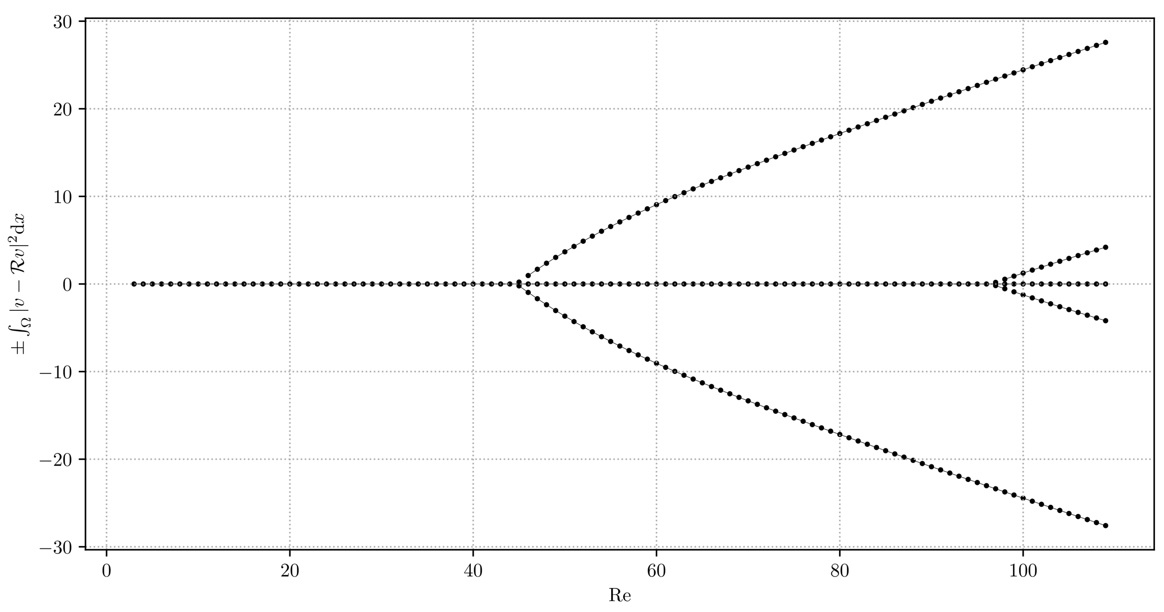

Figure 2: Bifurcation diagram, **FENE–CR**, $\mathrm{Wi} = 2, L^2 = 100, \beta = 0.5$

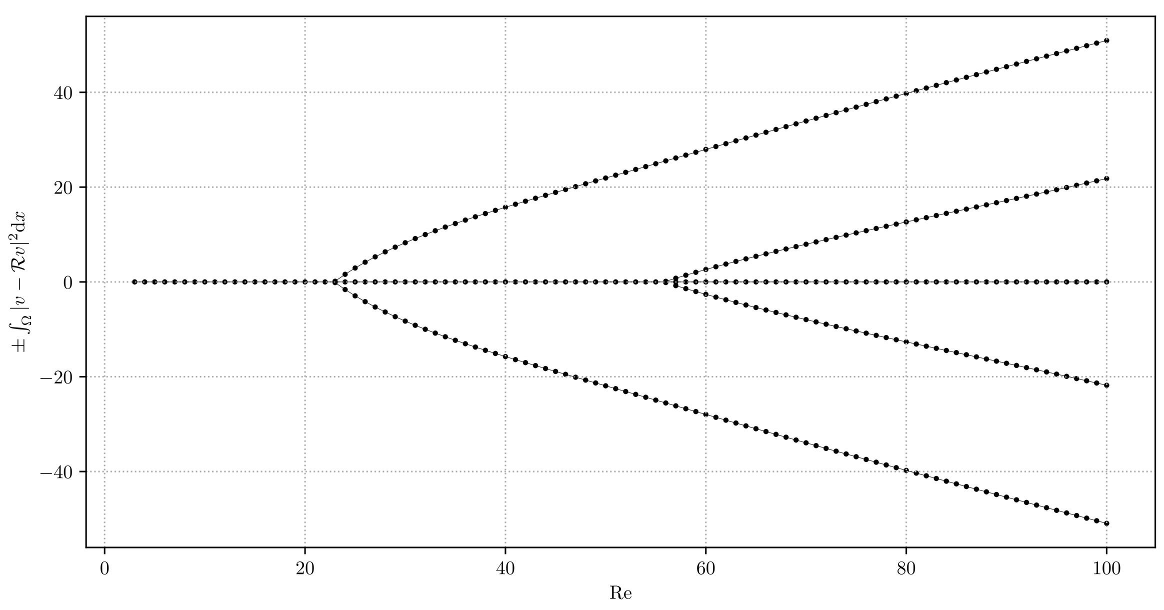

Figure 3: Bifurcation diagram, **Giesekus**, $\mathrm{Wi} = 5, \alpha = 0.5, \beta = 0.5$

Acknowledgements

This research was supported by the Czech Science Foundation grant number 201127X.

Contact

Ladislav Trnka, trnka.ladis@gmail.com